上海交通大學
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Some property of a max-type recursive model

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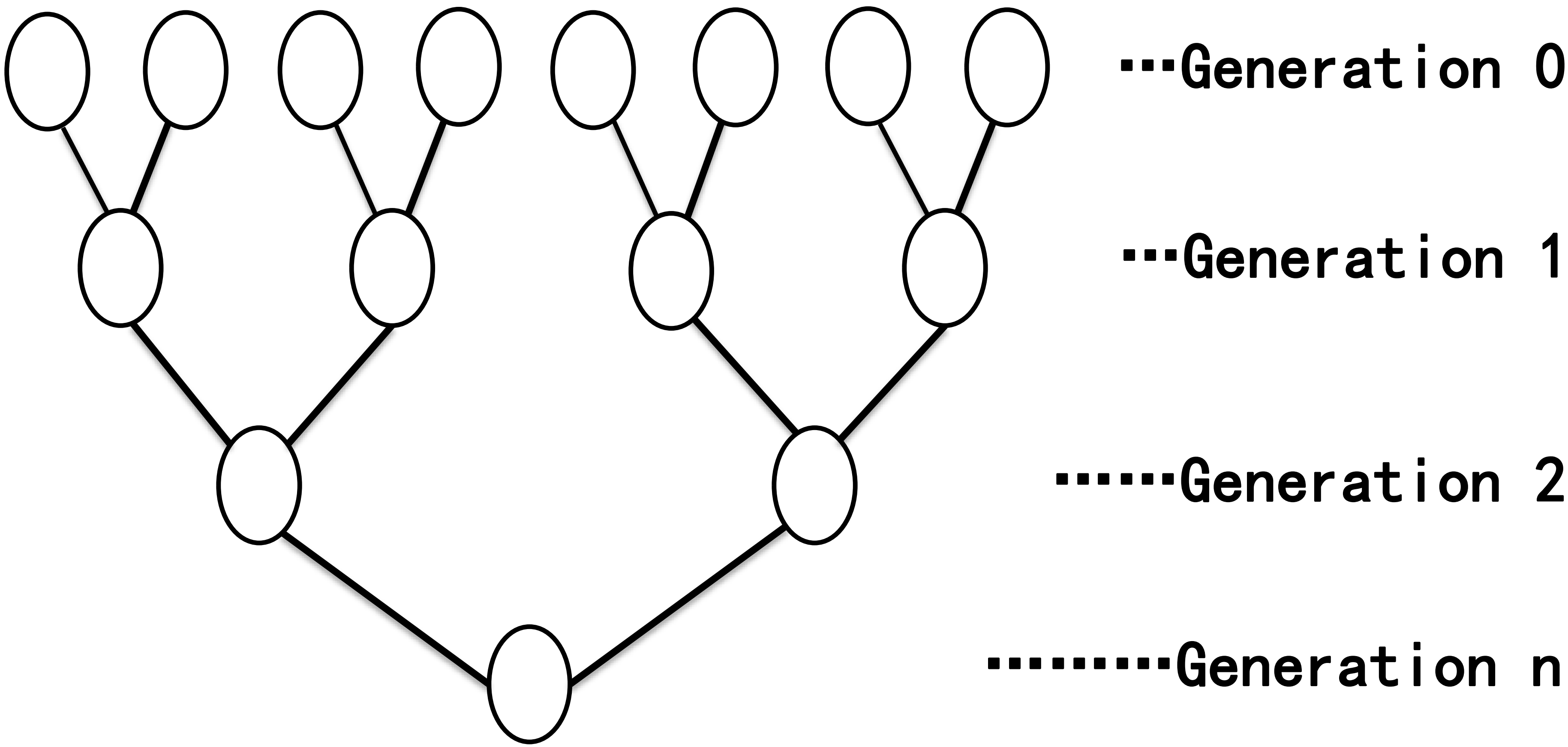
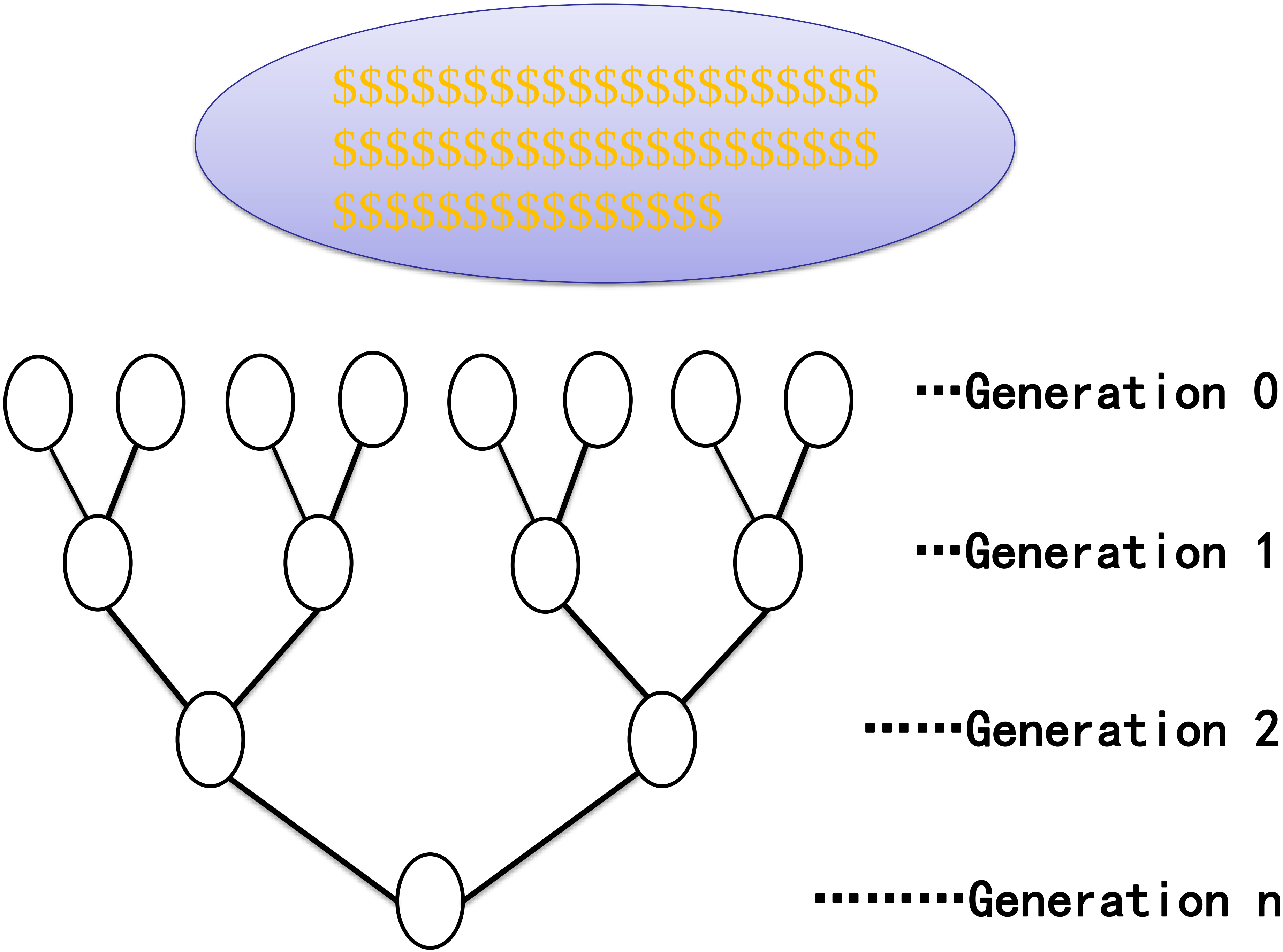
Workshop on Markov Processes and Related Topics
(SCU and BNU 2018)

Joint work with Dagard, Derrida, Hu, Lifshits, Shi





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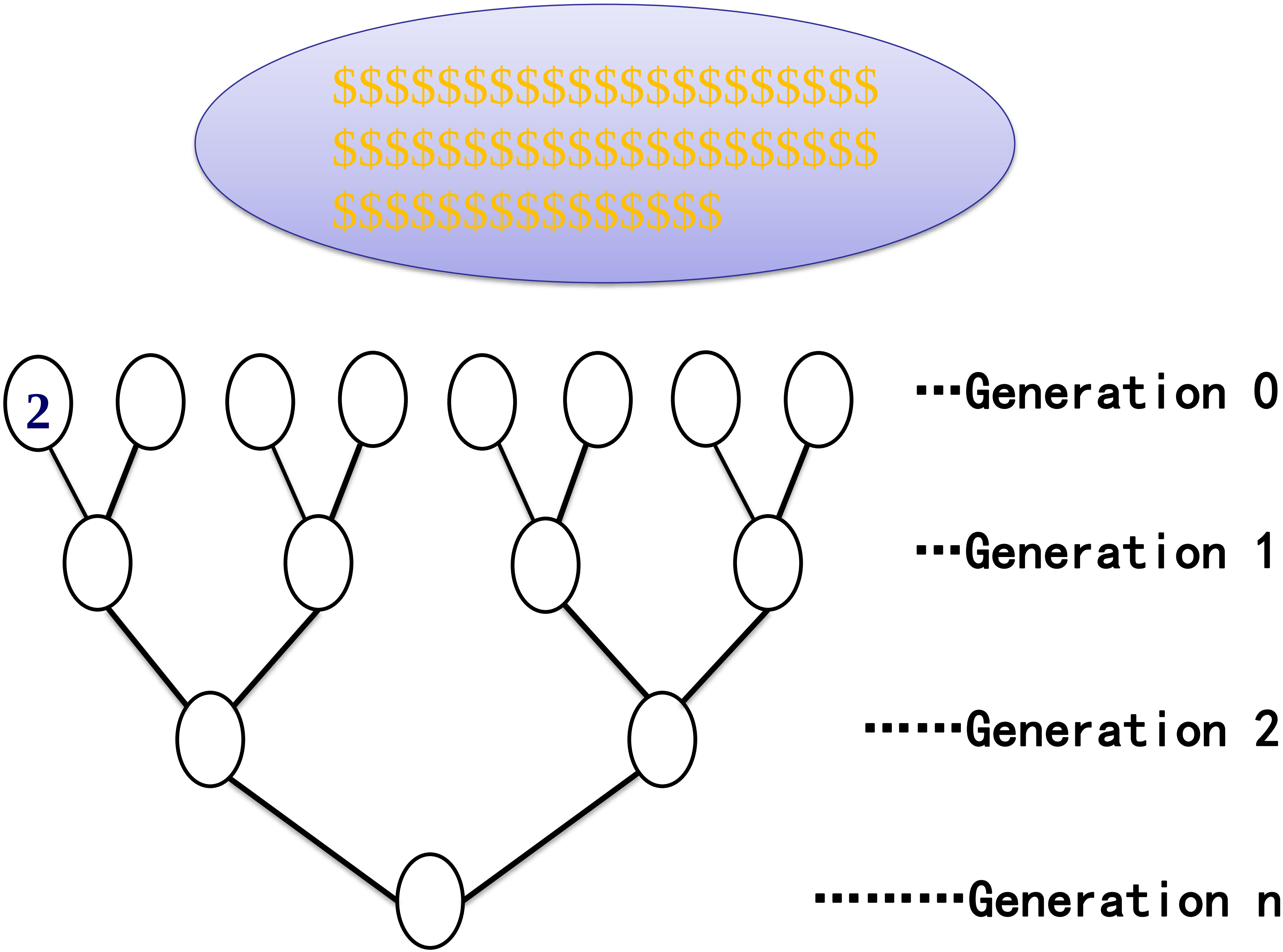
Generation n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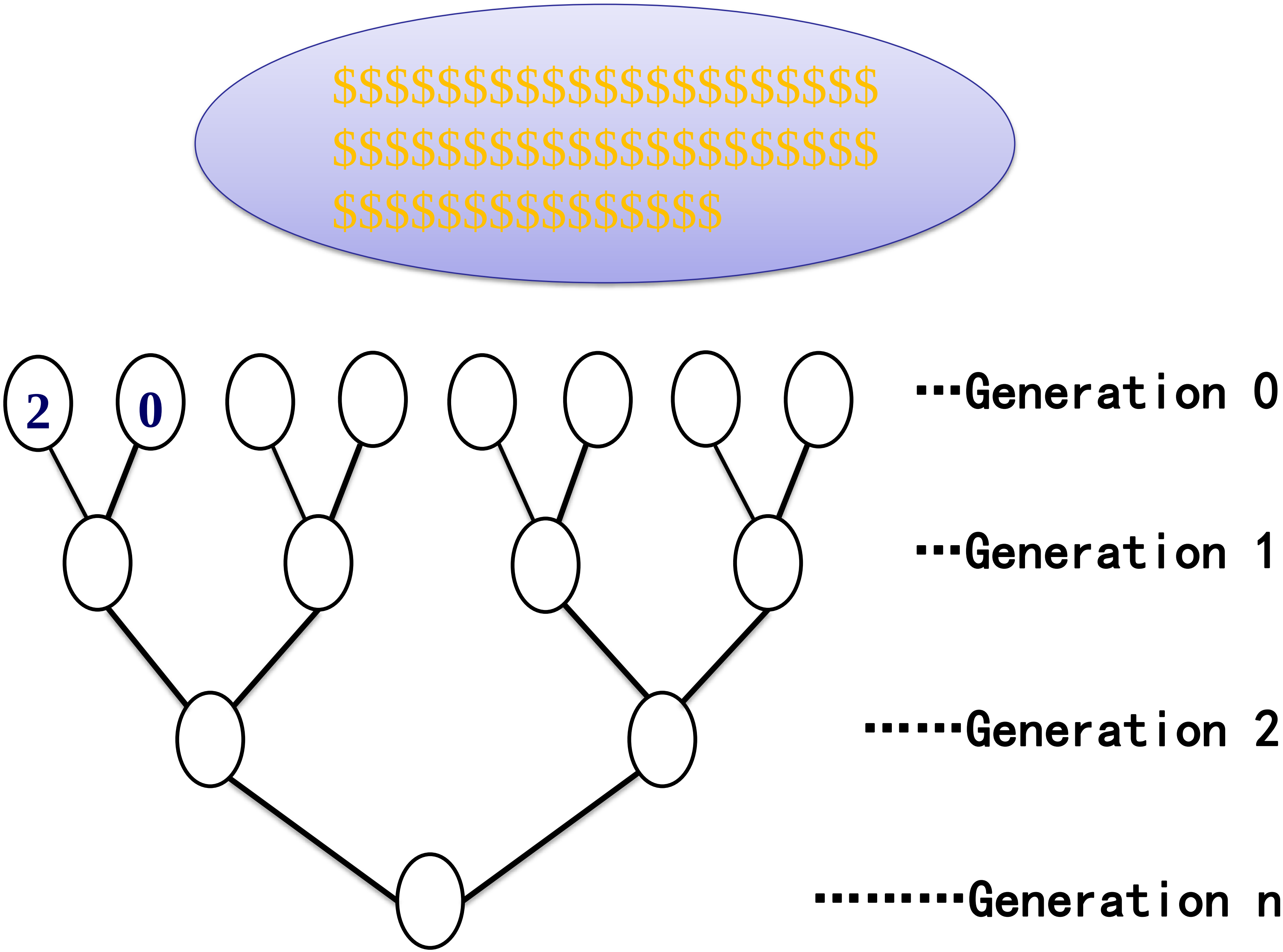
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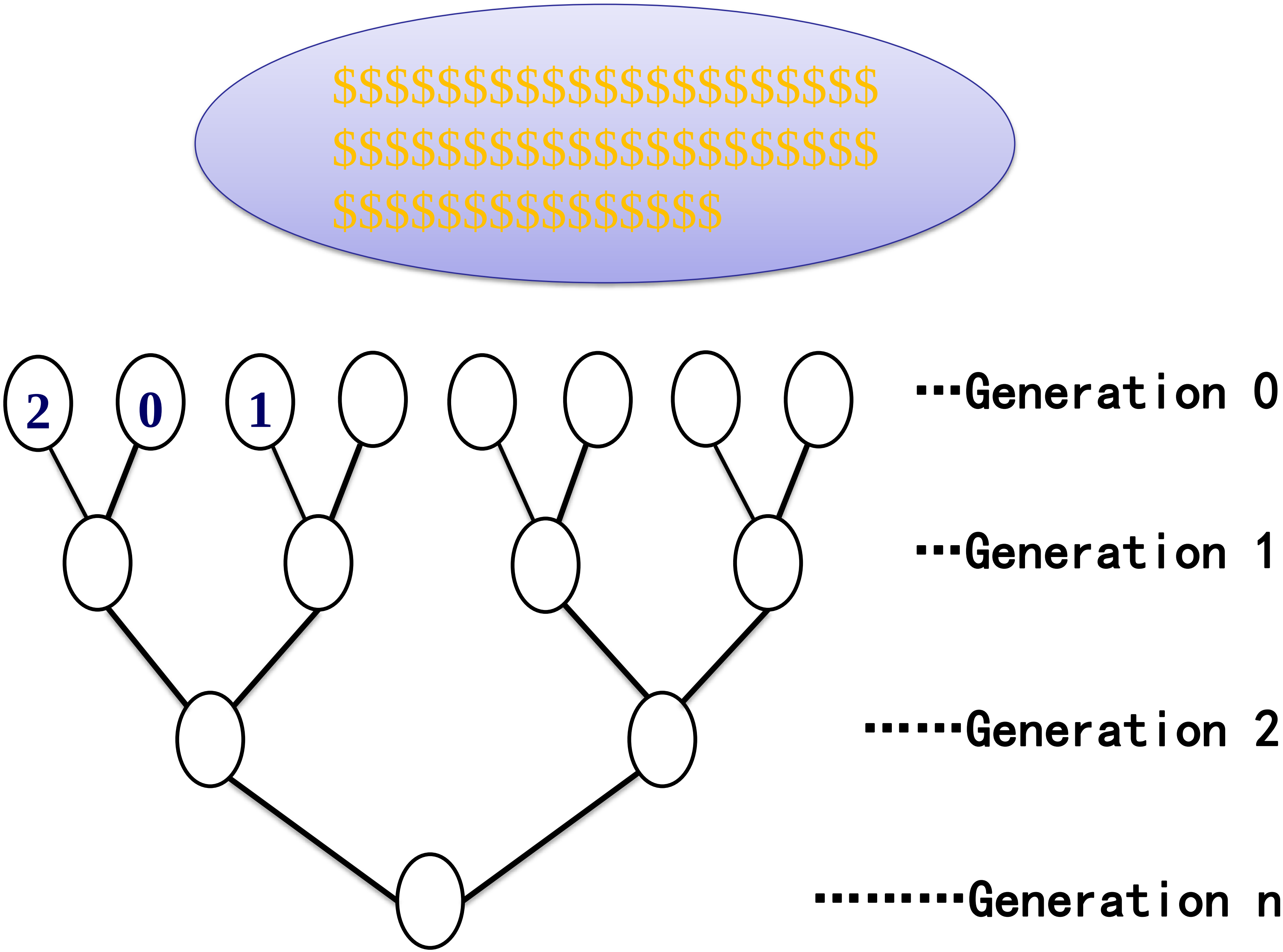

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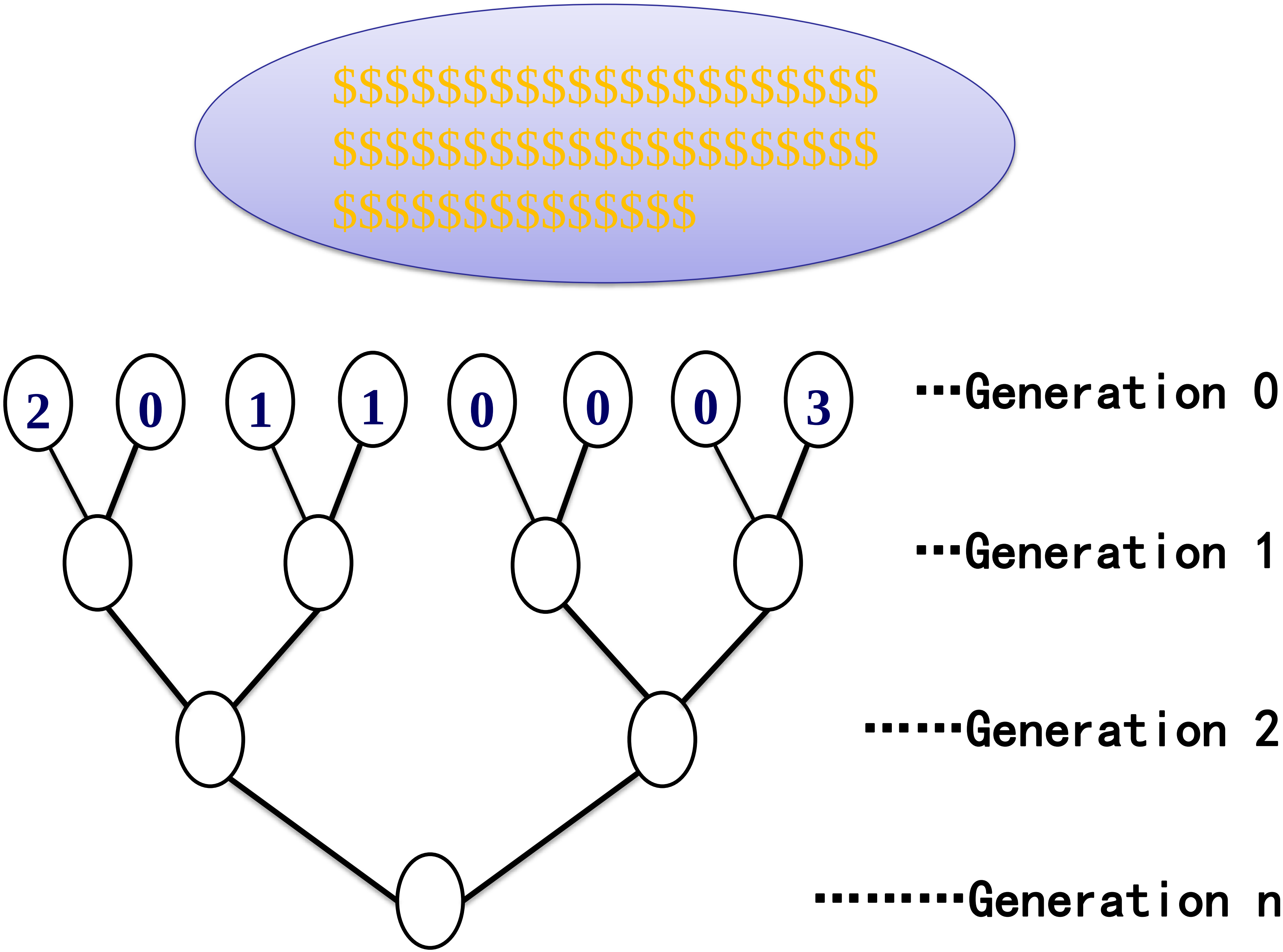

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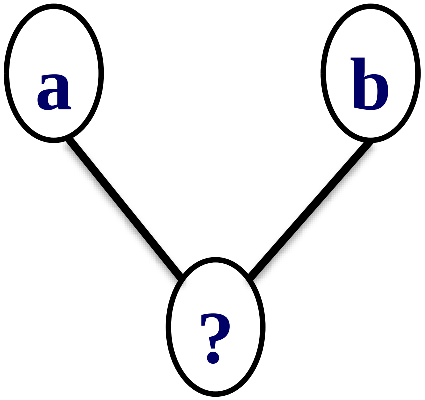
Suppose i-th person in generation 0: $x_0(i), i = 1, \cdots, 2^n$

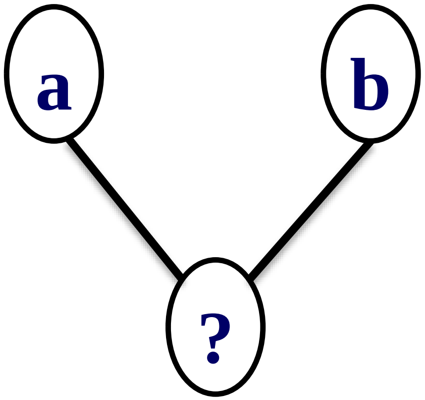


⋯Generation $k-1$

⋯Generation $k \geq 1$

Rule: $(a, b) \rightarrow ?$

Suppose i-th person in generation 0: $x_0(i), i = 1, \cdots, 2^n$

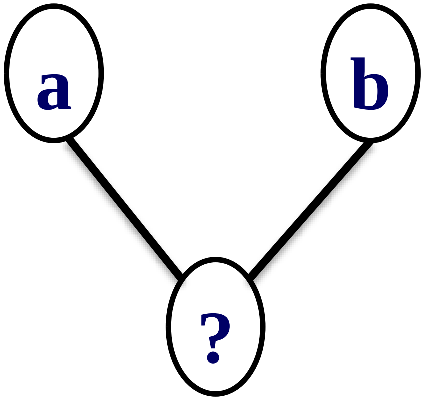


$\cdots$Generation $\boldsymbol{k-1}$

$\cdots$Generation $\boldsymbol{k \geq 1}$

Rule: $(\boldsymbol{a}, \boldsymbol{b}) \rightarrow \boldsymbol{a} + \boldsymbol{b}$

Suppose i-th person in generation 0: $x_0(i), i = 1, \cdots, 2^n$

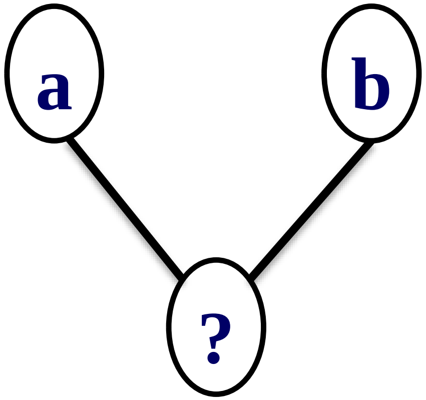


⋯Generation $k-1$

⋯Generation $k \geq 1$

Rule: $(a, b) \to a + b - 1$

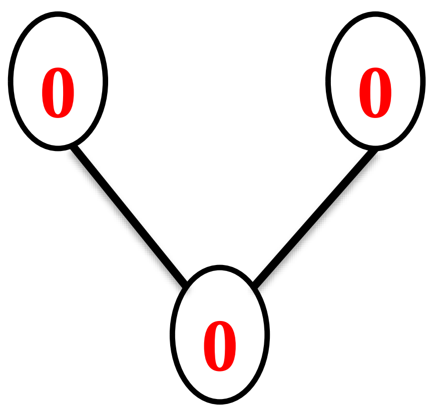
Suppose i-th person in generation 0: $x_0(i), i = 1, \cdots, 2^n$

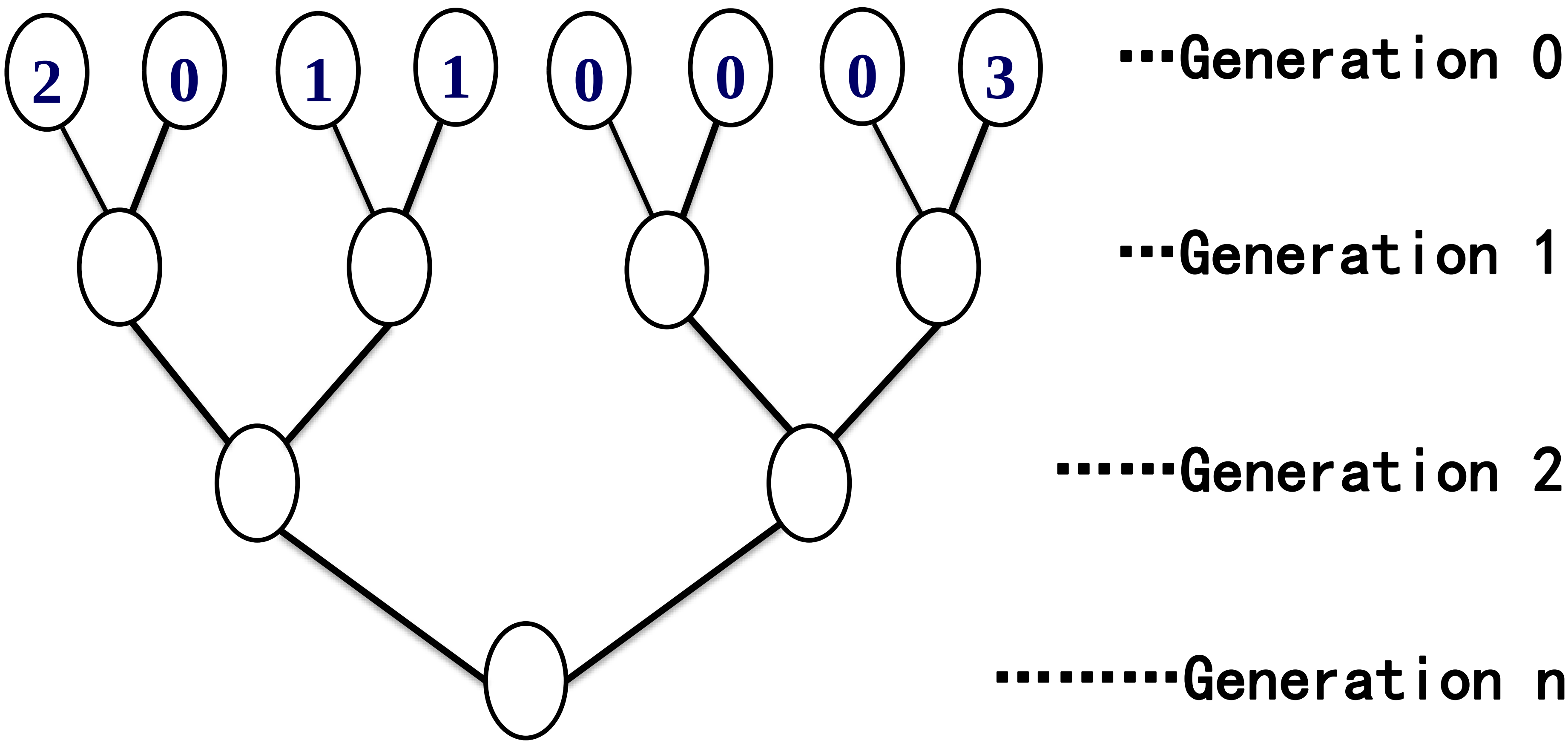


$\cdots$Generation $k-1$

$\cdots$Generation $k \geq 1$

Rule: $(a, b) \rightarrow (a+b-1)^+ = max\{a+b-1, 0\}$

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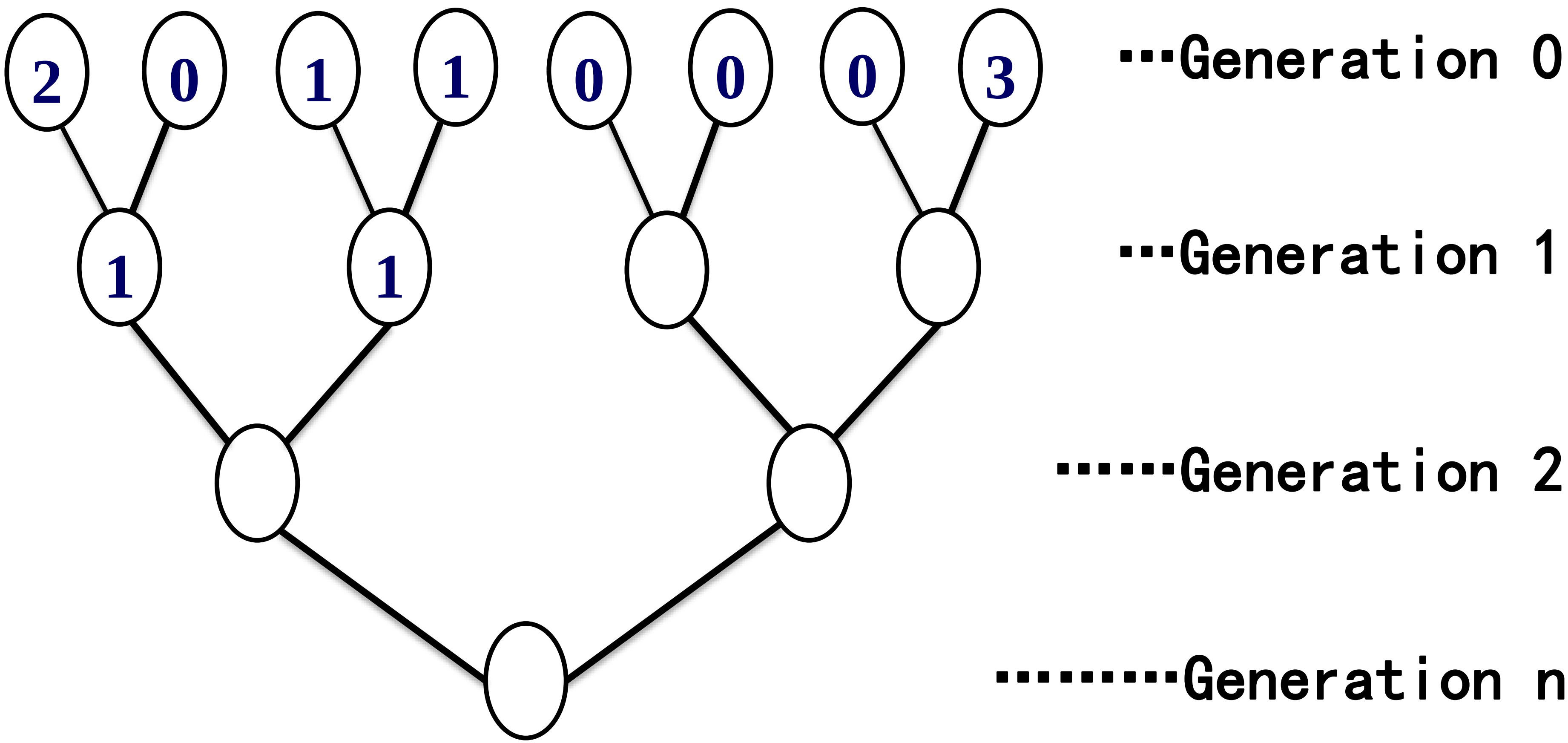
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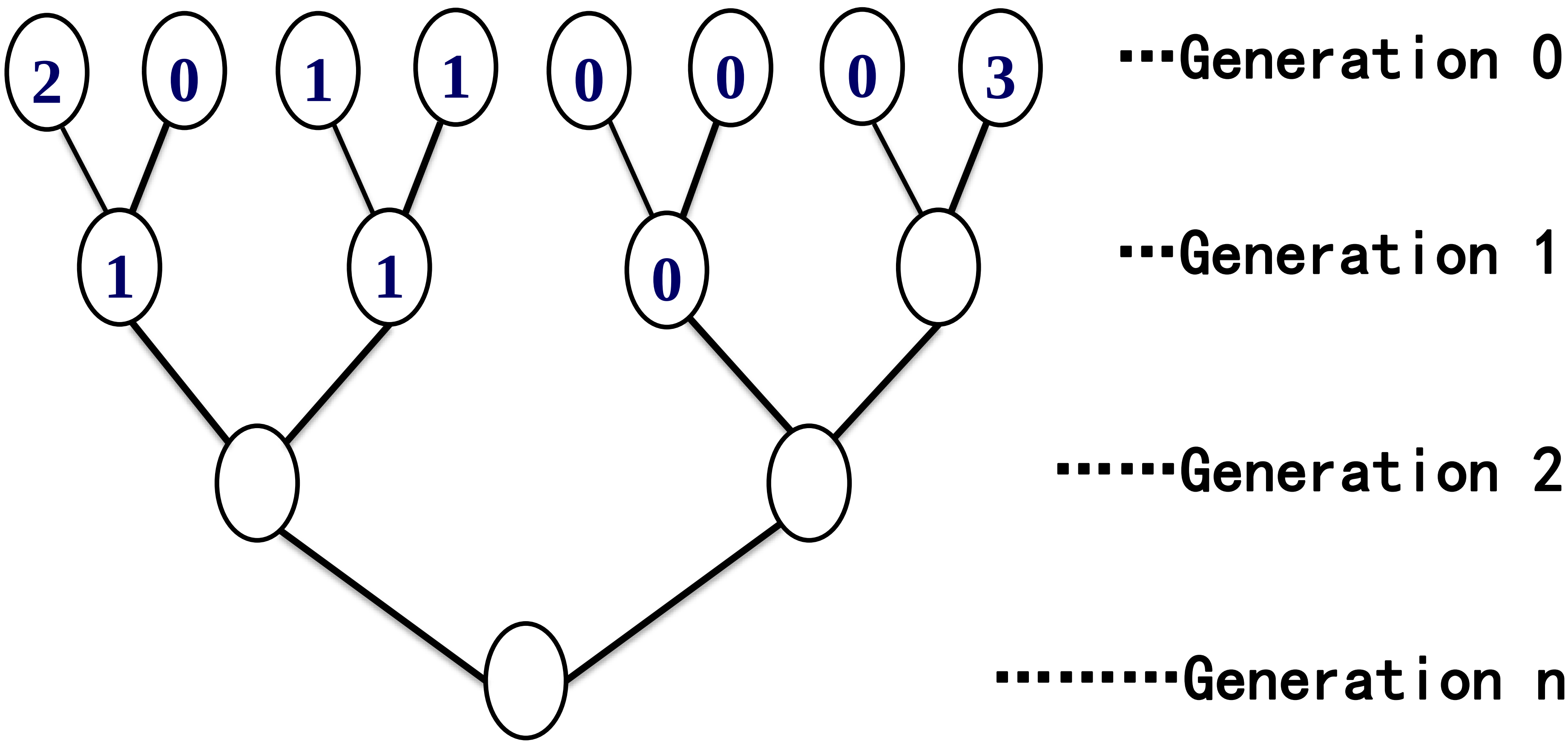
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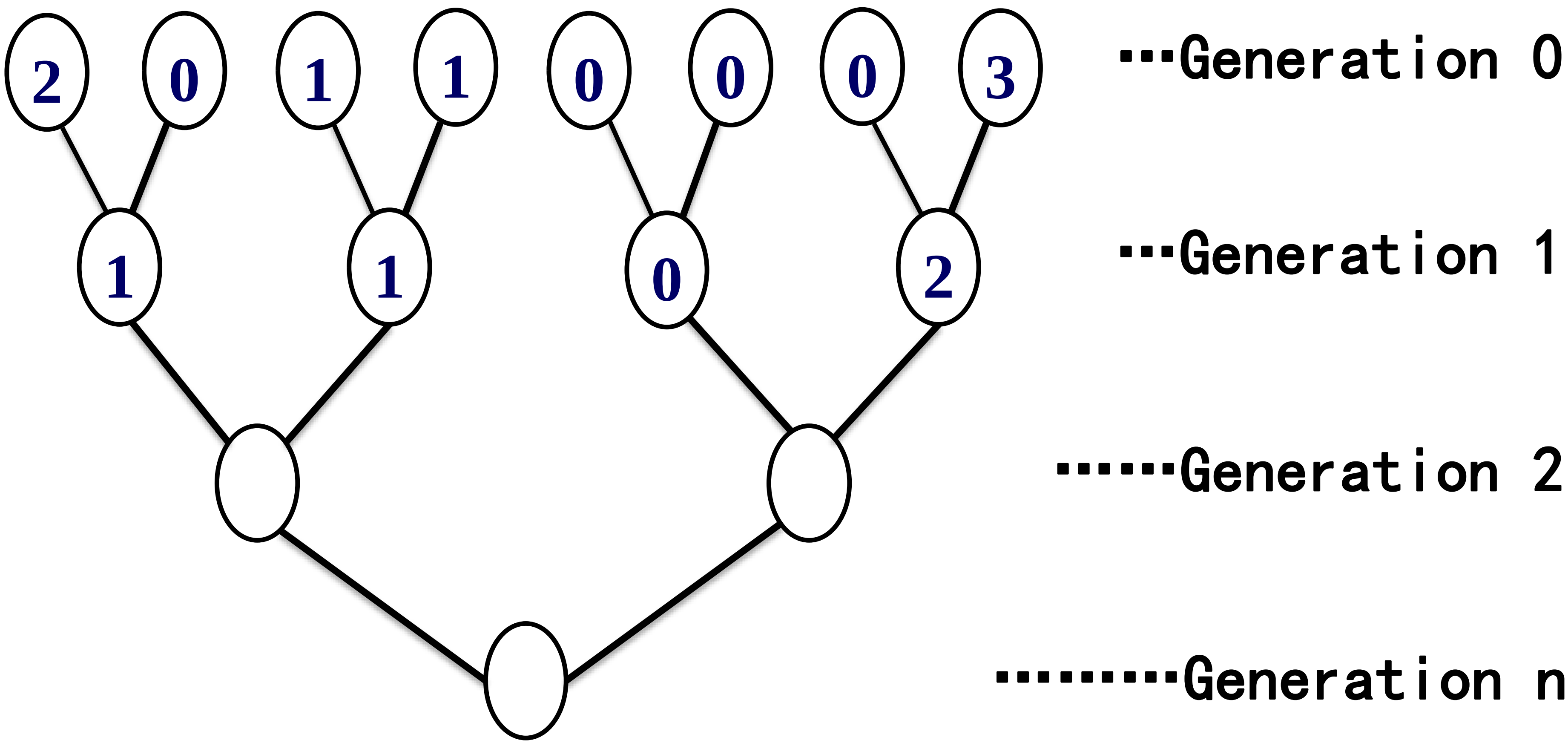
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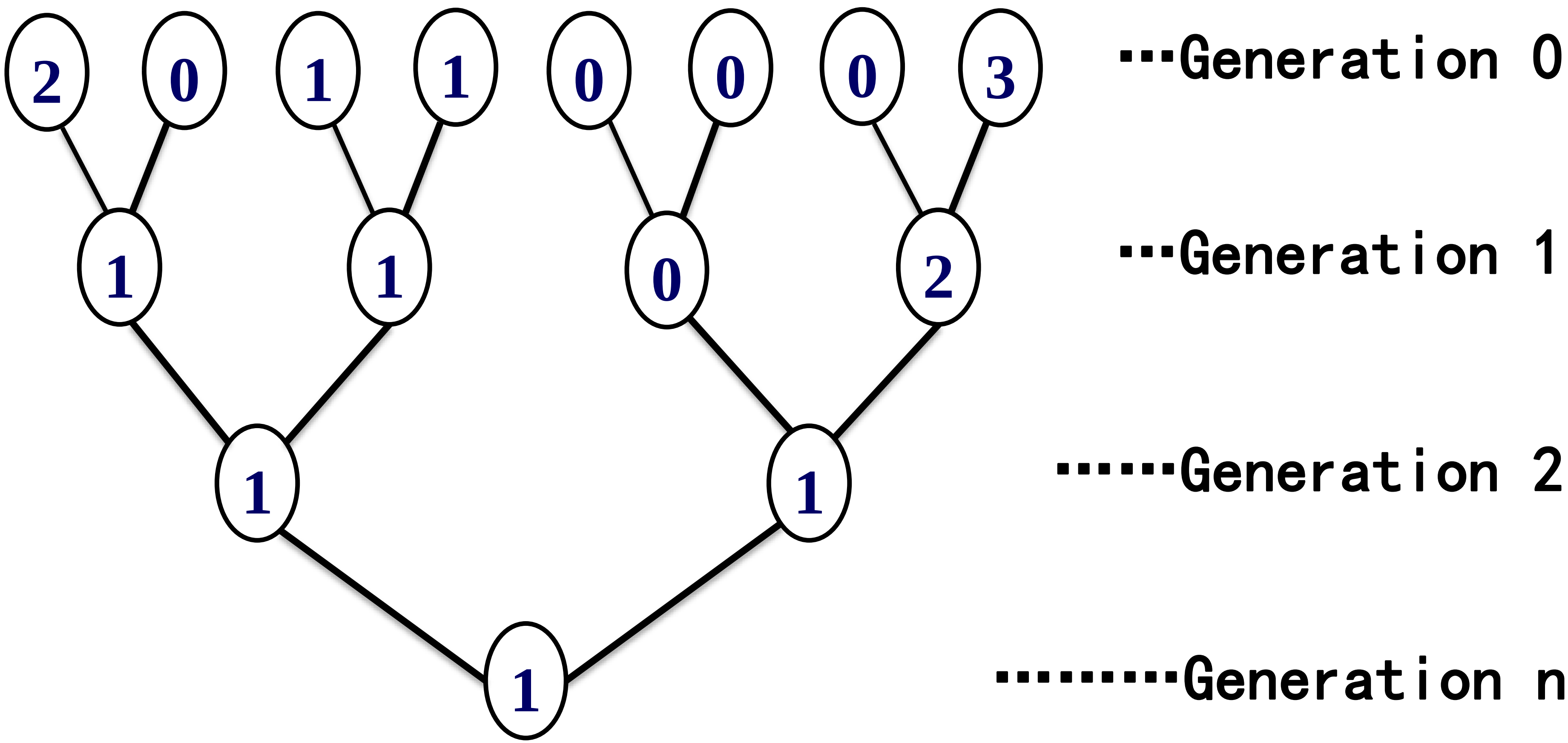
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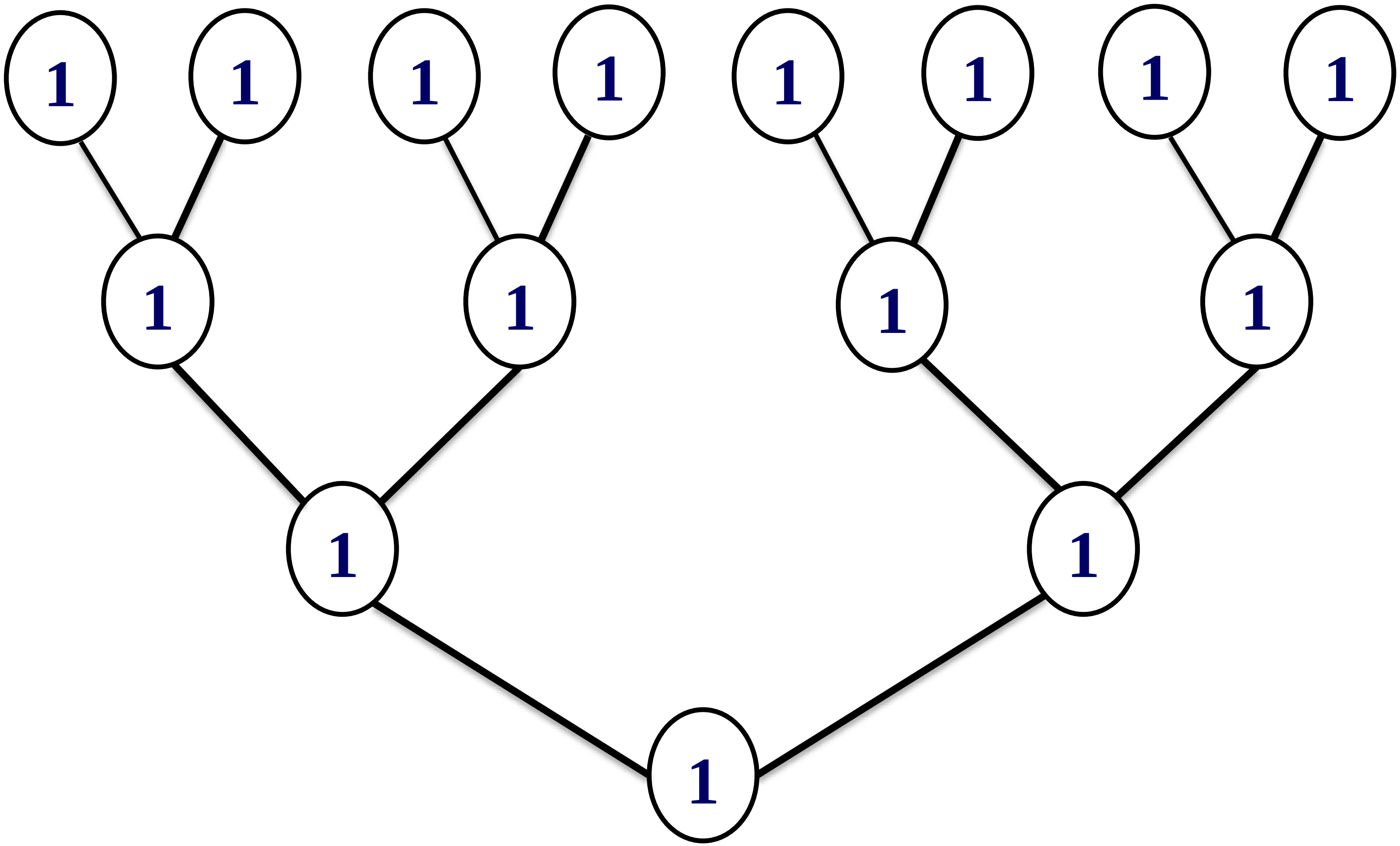
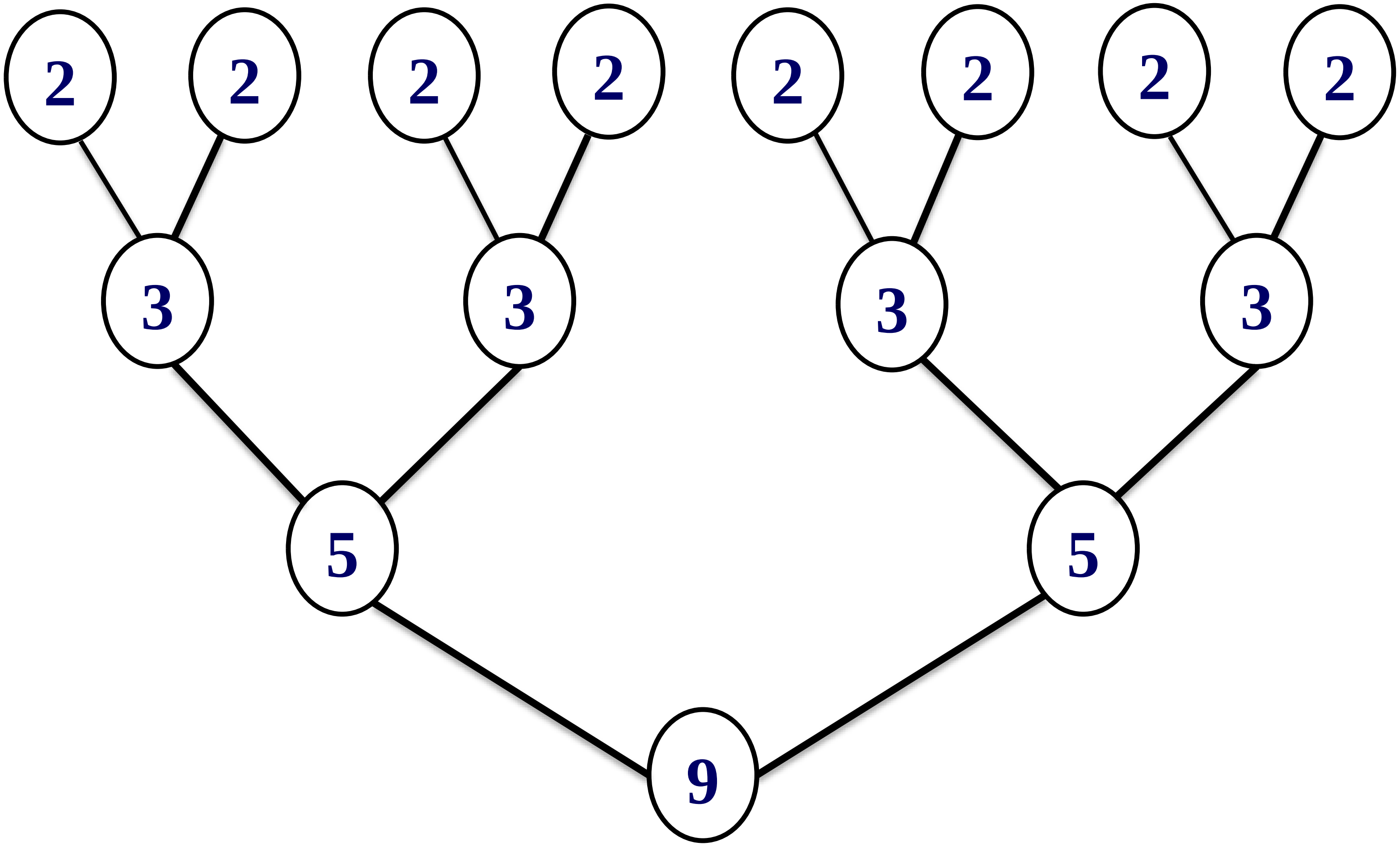
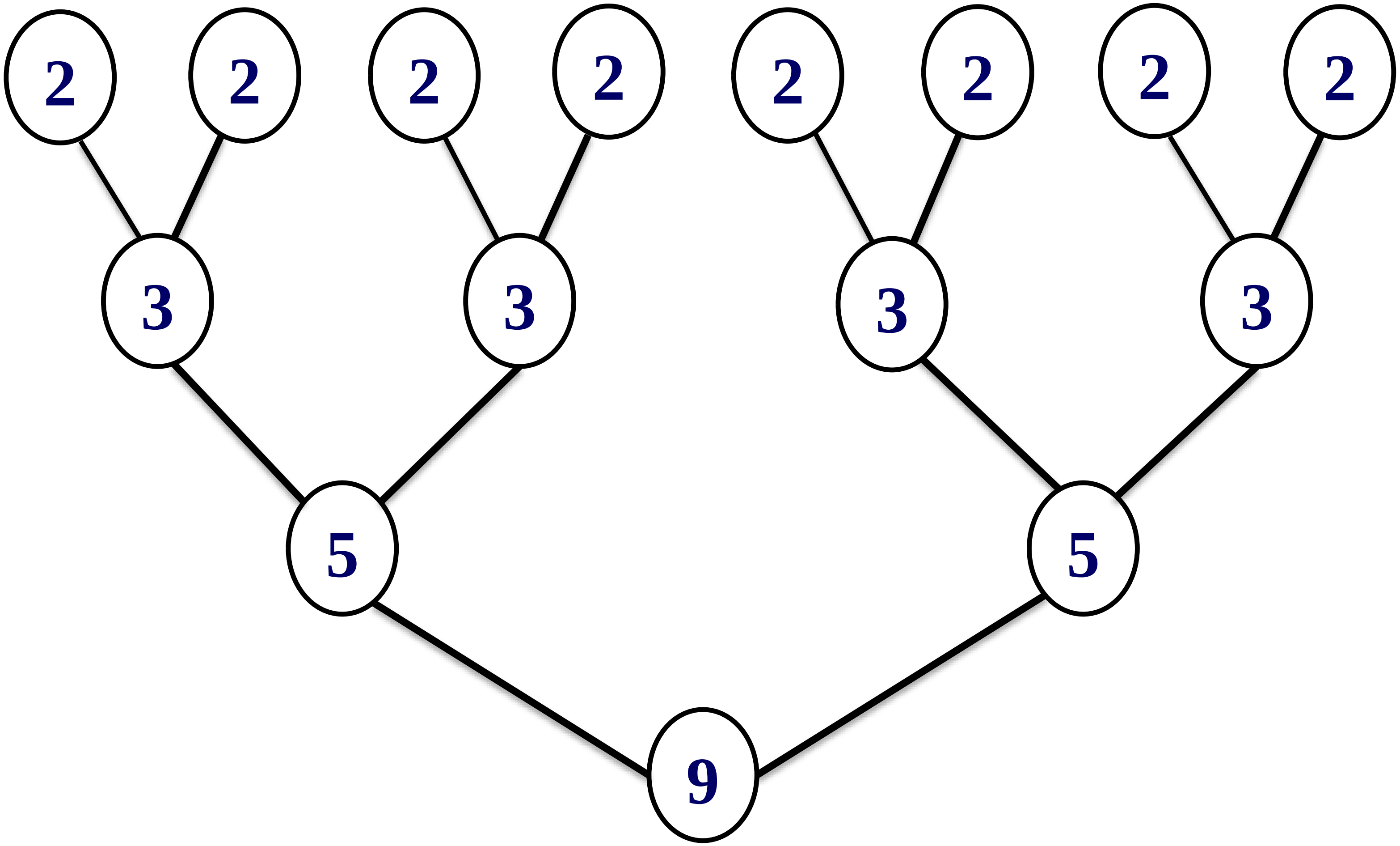
If $x_0(i) \equiv 1$ then $x_n = 1$



If $x_0(i) \equiv 2$ then $x_n = 2^n + 1$



If $x_0(i) \equiv 2$ then $x_n = 2^n + 1$



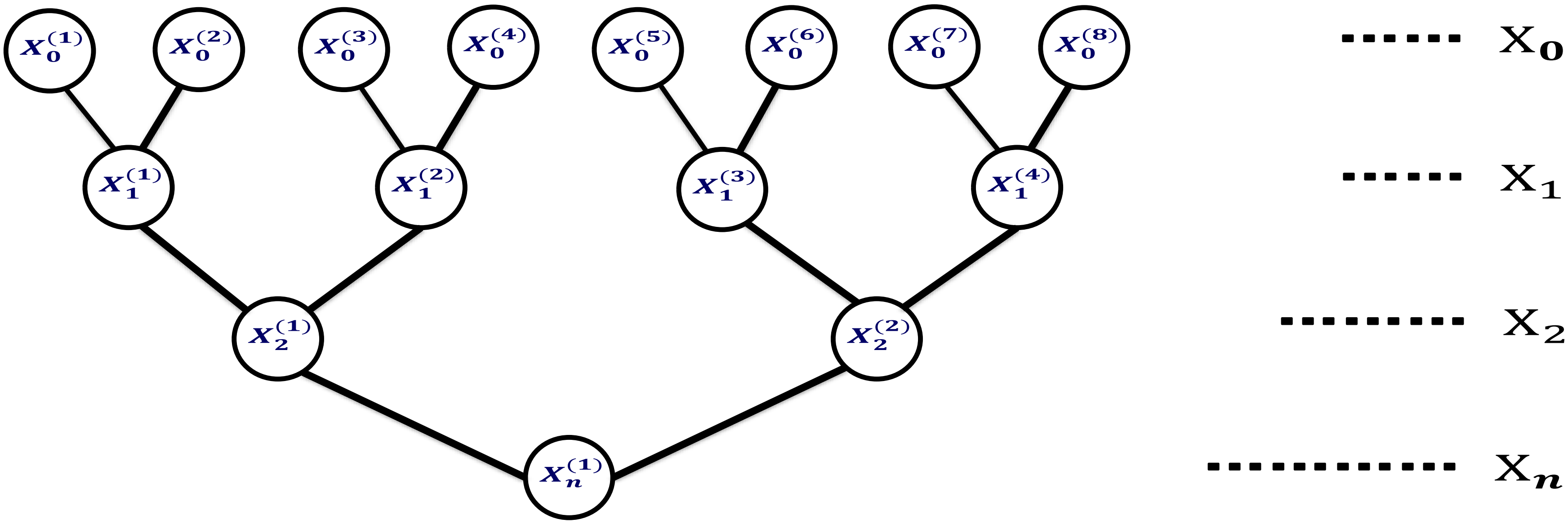
If $x_0(i) \equiv k \geq 2$ then $x_n = (k-1)2^n+1$

Next, we consider stochastic model

Let $p \in [0,1]$, $a_l \geq 0$ with $\sum_1^\infty a_l = 1$

Assume $X_0(i), i = 1, \dots, 2^n$, i.i.d. $\sim X_0$

$P(X_0 = 0) = 1 - p, \qquad P(X_0 = l) = p a_l, \qquad l \geq 1.$



Then the inherit rule says

$$X_{k+1} = max\{X_k^{(1)} + X_k^{(2)} - 1,\ 0\}$$

i.i.d. $\sim X_k$

One expects that

$$\lim_{n\to\infty} \frac{\boldsymbol{E(X_n)}}{\boldsymbol{2^n}} \ \textit{exists}$$

and there exists $\boldsymbol{p_c} \in (\boldsymbol{0}, \boldsymbol{1})$ *such that*

$$\lim_{n\to\infty} \frac{\boldsymbol{E(X_n)}}{\boldsymbol{2^n}} \begin{cases} > \boldsymbol{0}, & \boldsymbol{p} > \boldsymbol{p_c} \\ = \boldsymbol{0}, & \boldsymbol{p} < \boldsymbol{p_c} \end{cases}$$

Theorem 1 (Collet-Eckmann-Glaser-Martin 1984)

$$p_c: \quad E\left(2^{X_0}\right) = E\left(X_0 2^{X_0}\right)$$

$$\Rightarrow \qquad \frac{1}{p_c} = 1 + \sum_{2}^{\infty} (l-1) 2^l a_l$$

So, if $\begin{cases} a_1 \neq 1 \\ \sum l 2^l a^l < \infty \end{cases}$ then $p_c \in (0,1)$

$F_\infty(X) \coloneqq \lim_{n\to\infty} \frac{E(X_n)}{2^n}$, Free energy

$$F_\infty(p) = F_\infty(X) \coloneqq \lim_{n\to\infty} \frac{E(X_n)}{2^n},$$ Free energy

$$F_\infty(p) = F_\infty(X) := \lim_{n\to\infty} \frac{E(X_n)}{2^n},$$ Free energy

Conjecture 2. (Derrida-Retaux 2014)

If $\boldsymbol{p_c} \in (\mathbf{0}, \mathbf{1})$,

$$\boldsymbol{F_\infty(p_c + \varepsilon) = exp\left(-\frac{K + o(1)}{\varepsilon^{1/2}}\right), \qquad \varepsilon \downarrow 0}$$

3. Our resutls

Theorem 3

$$\mathbf{p} = p_c \quad \Rightarrow \quad \lim_{n\to\infty} E(X_n) = 0$$

Under some moment condition,

$$\mathbf{p} > p_c \quad \Rightarrow \quad CLT$$

$$\mathbf{p} < p_c \quad \Rightarrow \quad E(X_n) < e^{-\alpha n}$$

A weak version of Derrida-Retaux conjecture

Theorem 4

$$\exp\left(-\frac{K_2 \ln(\frac{1}{\varepsilon})}{\varepsilon^{1/2}}\right) \leq F_\infty(p_c+\varepsilon) \leq \exp\left(-\frac{K_1}{\varepsilon^{1/2}}\right)$$

A weak version of Derrida-Retaux conjecture

Theorem 4

$$exp\left(-\frac{K_2 ln(\frac{1}{\varepsilon})}{\varepsilon^{1/2}}\right) \leq F_\infty(p_c+\varepsilon) \leq exp\left(-\frac{K_1}{\varepsilon^{1/2}}\right)$$

Conjecture 5

$$p = p_c \qquad \Rightarrow \qquad E(X_n) \sim \frac{8}{n^2}$$

known $\quad sup\, nE(X_n) < \infty$

References:

[1]Chen,Derrida,Hu,Lifshits,Shi(2017).A max-type recursive model: some properties and open questions. arXiv:1705.04787

[2]Collet,Eckmann,Glaser,Martin (1984).Study of the iterations of a mapping associated to a spin-glass model.Commun.Math.Phys.94,353--370.

[3] Derrida,Retaux(2014).The depinning transition in presence of disorder: a toy model, J.Statist.Phys.156,268--290.

Thank you for your attention !